Optimal stopping for Hunt and Lévy processes

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Plan of the talk

1. Motivation: from Finance and Mathematics

2. Preliminaries on Hunt processes

3. Main Theorem: Optimal stopping through Green kernels

4. Application to Lévy processes

5. Closed solution for the distribution of the maximum.

6. A simple example: Complete solution for compound Poisson process

7. Some open questions and remarks.

1. Financial motivation:

Price a *perpetual american call option*:

$$S_t = S_0 \exp(X_t) \qquad g_c(x) = (x - K)^+$$

where

- $\{X_t\}$ is a *stochastic process*
- $g_c(x)$ is the *payoff function*

Optimal stopping problem: Find

- the *value function* $V(x)$
- the *optimal stopping time* τ^*

such that

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x \left(\mathrm{e}^{-r\tau} g(X_\tau) \right) = \mathbf{E}_x \left(\mathrm{e}^{-r\tau^*} g(X_{\tau^*}) \right).$$

Our purpose:

- Consider processes $\{X_t\}$ as general as possible
- Consider functions $g(x)$ as general as possible
- But: Obtain closed or explicit solutions, or as explicit as possible.

1. Mathematical motivation:

If $\{X_t\}$ is a Lévy process, and

$$M = \sup\{X_t \colon 0 \leq t \leq \tau(r)\}$$

where $\tau(r)$ is an independent exponential time, with parameter $r \geq 0$, the solution for g_c (similar for g_p) is[2]:

$$\tau^* = \inf\{t \geq 0 \colon X_t \geq x^* := K\,\mathbf{E}\exp(M)\}.$$

$$V(x) = \frac{\mathbf{E}\left(e^{x+M} - K\,\mathbf{E}(e^M)\right)^+}{\mathbf{E}(e^M)}.$$

- In words: Find the distribution of M to price the option.

[2] E.M., Finance and Stochastics (2002)

Similar results (solution in terms of M):

- For Lévy processes in Bachelier model, i.e.

$$S_t = x + X_t$$

  and g_p or g_c,

- For $g(x) = (x^+)^n$ and random walks (Novikov and Shiryaev (2004)),

- For $g(x) = (x^+)^n$ and Lévy processes (Kyprianou and Surya (2005)),

- For $g(x) = (x^+)^a$ ($a > 1$ real) and Lévy processes (Novikov and Shiryaev (2006))

- For general g and Regular exponential Lévy processes (Boyarchenko and Levendorskii, 2002)

Main questions

- Why does M appear in the solution?

- Can you find a probabilistic or analytic explanation?

- A first answer to this question, in a sub-class of Lévy process, was found by Boyarchenko and Levendorskii (2002). They worked analytically, with pseudo-differential operators.

- In this talk we present a second answer, based on the general theory of Markov processes, with probabilistic arguments.

2. Transient Hunt processes:

$X = \{X_t\}$ is a strong Markov process, quasi left continuous with paths right continuous with left limits with $X_0 = x$.

The resolvent or Green Kernel is

$$G(x, A) := \int_0^\infty \mathrm{e}^{-rt}\, \mathbf{P}_x(X_t \in A)\, dt,$$

where $x \in \mathbf{R}$ and A is a Borel subset of $\mathbf{R}$. Assume that $G(x, A)$ is absolutely continuous w.r.t Lebesgue measure, and denote $G(x, y)$ the Radon-Nykodim density

A function $v : \mathbf{R} \mapsto [0, +\infty]$ is r-excessive if

$$\mathbf{E}_x\, \mathrm{e}^{-r\tau}(v(X_\tau)) \leq v(x) \qquad \text{for all stopping times } \tau$$

Examples of Hunt processes in finance:

$$S_t = S_0 \exp(X_t),$$

where

- $X_t = \sigma W_t + \left(r - \sigma^2/2\right)t$ (Black-Scholes Model)
- $\{X_t\}$ is a diffusion (Local volatility model)
- $\{X_t\}$ is a Lévy process
- Local volatility model with jumps:

$$dX_t = a(b - X_t)dt + \sigma(X_t)dJ_t$$

  where J_t is a Lévy process (Mean reverting process with jumps)

Three key results:

A. The value function $V(x)$ is an excessive function s.t.

$$g(x) \leq V(x) \leq f(x)$$

for any other $f(x)$ excessive. This is the majorant and least excessive characterization of the value function by Dynkin (1963).

B. The function $v(x)$ is excessive $\Leftrightarrow$ $\exists$ a radon measure $\sigma(dy)$ s.t.

$$v(x) = \int_{\mathbf{R}} G(x, y)\, \sigma(dy) \tag{1}$$

C. Theorem: If $\sigma(N) = 0$, then v is harmonic for X on the set N:

For $\tau_N = \inf\{t \geq 0 \colon X_t \notin N\}$ we have

$$v(x) = \mathbf{E}\left(\mathrm{e}^{-r\tau_N}\, v(X_{\tau_N})\right),$$

Application to optimal stopping:

- If $\sigma(N) = 0$ we do not loose value inside N.
- The Null set is the Non-stopping region.
- The complement of N is the Support of σ, also the Stopping region.

3. Main Theorem.

- $\{X_t\}$ is a Hunt process,
- $g(x) \geq 0$ is a Borel function, and
- $r \geq 0$ is discount factor s.t.

$$\mathbf{E}_x \left[\sup_{t \geq 0} e^{-rt} g(X_t) \right] < \infty.$$

Denote

$$V(x) := \int_{[x^*,\infty)} G(x,y)\sigma(dy)$$

with a Radon measure $\sigma(dy)$ with support on $S = [x^*, \infty)$.

Assume that:

(a) V is continuous,

(b) $V(x) \to 0$ when $x \to -\infty$.

(c) $V(x) = g(x)$ when $x \geq x^*$ (in S: stop),

(d) $V(x) \geq g(x)$ when $x < x^*$ (in N: non-stop).

Then the solution to the optimal stopping problem for $\{X_t\}$ and $g(x)$ is

$$\tau^* = \inf\{t \geq 0\colon X_t \geq x^*\}.$$

is an optimal stopping time and $V(x)$ is the value function:

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x \left(\mathrm{e}^{-r\tau} g(X_\tau)\right) = \mathbf{E}_x \left(\mathrm{e}^{-r\tau^*} g(X_{\tau^*})\right).$$

Some remarks

- More general results can be obtained for more general Supports (two sided, etc.)

- V is excessive by construction.

- $N = (-\infty, x^*)$ is the non-stopping or continuation region as $\sigma(N) = 0$.

- Consequently $S = [x^*, \infty)$ is the stopping region.

- Difficult part is (c): find $\sigma(dy)$ s.t. $V(x) = g(x)$ on S.

4. Application: Lévy processes and maxima.

A Hunt process with independent and stationary increments is a *Lévy process*. If $v \in \mathbf{R}$, Lévy-Khinchine formula:

$$\mathbf{E}(\mathrm{e}^{ivX_t}) = \mathrm{e}^{t\psi(iv)}$$

where, for complex $z = iv$ the *characteristic exponent* is

$$\psi(z) = az + \frac{1}{2}b^2 z^2 + \int_{\mathbf{R}} \big(\mathrm{e}^{zx} - 1 - zh(x)\big)\Pi(dx).$$

Here

- the truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$ is fixed,
- a is the drift, σ the variance of the Gaussian part,
- Π is the jump measure

Wiener - Hopf factorization

Remember that $\tau(r)$ an exponential time, with parameter r, independent of X.

Denote

$$M = \sup_{0 \le t < \tau(r)} X_t \qquad \text{and} \qquad I = \inf_{0 \le t < \tau(r)} X_t$$

called the *supremum* and the *infimum* of the process.

Wiener-Hopf-Rogozin factorization states

$$\mathbf{E}(\mathrm{e}^{ivX_{\tau(r)}}) = \frac{r}{r - \psi(iv)} = \mathbf{E}(\mathrm{e}^{ivM})\,\mathbf{E}(\mathrm{e}^{ivI})$$

Theorem II: Lévy Processes.

Assume Main Theorem hold for X a Lévy process. Then, there exists a function $Q\colon [x^*, \infty) \to \mathbf{R}$ s.t.

$$V(x) = \mathbf{E}\left(Q(x+M)\,;\, x+M \geq x^*\right), \qquad x \leq x^*.$$

Example: For the call option

$$V(x) = \frac{\mathbf{E}\left(e^{x+M} - K\,\mathbf{E}(e^M)\right)^+}{\mathbf{E}(e^M)}$$

gives

$$Q(y) = \frac{e^y - K\,\mathbf{E}(e^M)}{\mathbf{E}(e^M)}.$$

Main idea of the Proof

WH factorization can be written as

$$X_{\tau(r)} = M + \tilde{I}$$

where M and $\tilde{I}$ are *independent*

Furthermore

$$rG(x, dy) = \mathbf{P}_x(X_{\tau(r)} \in dy),$$

and:

$$rG(x,y) = \begin{cases} \int_{-\infty}^{y-x} f_I(t) f_M(y-x-t)dt, & \text{if } y-x<0, \\ \\ \int_{y-x}^{\infty} f_M(t) f_I(y-x-t)dt, & \text{if } y-x>0. \end{cases}$$

If we plugg in this formula in the Main Theorem

$$
\begin{aligned}
V(x) &= \int_{x^*}^{\infty} G(x,y)\sigma(dy) \\
&= \dots \\
&= \dots \\
&= \int_{x^*-x}^{\infty} f_M(t)Q(x+t)dt = \mathbf{E}\left(Q(x+M)\,;\, x+M \geq x^*\right),
\end{aligned}
$$

where, for $z \geq x^*$, we denote

$$Q(z) = r^{-1}\int_{x^*}^{z} f_I(y-z)\sigma(dy).$$

This concludes the proof.

Closed solutions for $\mathbf{P}(M \le x)$

Consider a Lévy process with a and σ arbitrary and jump measure

$$\Pi(dx) = \begin{cases} \lambda p(x)dx & \text{if} \qquad x > 0. \\ \\ \Pi^{-}(dx) & \text{abitrary, if } x < 0, \end{cases}$$

where

$$p(x) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} c_{kj} (\alpha_k)^j \frac{x^{j-1}}{(j-1)!} e^{-\alpha_k x}, \quad x > 0,$$

is the density of a positive random variable.

The characteristic exponent is

$$\psi(z) = \lambda\Big[\sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}\Big(\frac{\alpha_k}{\alpha_k - z}\Big)^j - 1\Big] + \psi^-(z)$$

Theorem[3] For the considered LP we have

$$\mathbf{E}\, e^{zM_q} = \prod_{k=1}^{n}\left(\frac{\alpha_k - z}{\alpha_k}\right)^{m_k}\prod_{j=1}^{N}\left(\frac{\beta_j}{\beta_j - z}\right)^{n_j},$$

where $\beta_1, \ldots, \beta_N$ are the roots of the equation $\psi(z) = r$ on $\Re(z) > 0$.

[3] A. Lewis, E. M. Wiener-Hopf factorization for Lévy processes with negative jumps with rational transforms. *J. of Appl. Probability, (2008)*

From this the density of M is

$$f_{M_q}(x) = \sum_{k=1}^{N} \sum_{j=1}^{n_k} d_{kj} \big(\beta_k\big)^j \frac{x^{j-1}}{(j-1)!} e^{-\beta_k x}, \quad x > 0,$$

6. A simple example: Compount Poisson process and power reward

$$X_t = at + \sum_{i=1}^{N_t} Y_i,$$

where

- $a < 0$
- (N_t) is a Poisson process with parameter λ,
- (Y_i) are i.i.d. $\exp(\alpha)$.
- $r = 0$ (no discounting)
- $g(x) = \max(x, 0)^\gamma$ with $\gamma > 1$.

Problem: find $V(x), \tau^*$ s.t.

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x \left(g(X_\tau) \right) = \mathbf{E}_x \left(g(X_{\tau^*}) \right),$$

The Green kernel is

$$G(x,0) = \begin{cases} A_2\, \mathrm{e}^{\rho\, x}\, dx, & x \le 0, \\ -A_1, & x > 0, \end{cases}$$

where

$$\rho := \alpha + \frac{\lambda}{a} > 0$$

and

$$A_1 = \frac{\alpha}{\lambda + a\alpha} < 0, \qquad A_2 = \frac{\lambda}{a(\lambda + a\alpha)} > 0.$$

Notice that $\rho > 0$ means that a.s. $\lim_{t\to\infty} X_t = -\infty$.

Aim: find $\sigma(y)$ and x^* s.t.

$$V(x) = \int_{[x^*,+\infty)} G(x,y)\sigma(y)dy$$

has properties (a), (b), (c) and (d) given in Main Theorem.

As we want $V(x) = x^\gamma$ for $x > x^*$:

$$-\sigma'(x) + \alpha\sigma(x) = a\gamma(\gamma - 1)x^{\gamma-2} - (a\alpha + \lambda)\,\gamma x^{\gamma-1}.$$

to give

$$\sigma(x) = -a\gamma\, x^{\gamma-1} - \lambda e^{\alpha x} \int_x^\infty e^{-\alpha y} \gamma\, y^{\gamma-1}\, dy.$$

Claim: the equation $\sigma(x) = 0$ has a unique root, that is

$$x^\gamma = \frac{\lambda}{(-a)}\, e^{\alpha x} \int_x^\infty e^{-\alpha z}\, z^\gamma\, dz,$$

has a unique positive root, denoted x^*.

The last step consist in the verification that the value function obtained is continuous and satisfies $V(x) > x^\gamma$ for $x < x^*$.

To conclude, the optimal stopping time τ^* is given by

$$\tau^* := \inf\{t : X_t \geq x^*\},$$

and the value function is

$$V(x) = \begin{cases} (x^*)^\gamma \exp\big(\rho(x - x^*)\big) & \text{for } x < x^*, \\ \\ x^\gamma & \text{for } x \geq x^\gamma. \end{cases}$$

Remark: Since $V'(x^*-) = \rho(x^*)^\gamma$ and $g'(x^*) = \gamma(x^*)^{\gamma-1}$ there is no smooth fit at x^*.

Numerical Results.

Main practical problem: find x^*.

- The computations are done with Mathematica-package
- Uses
  - a subroutine for incomplete gamma-function
  - programs for numerical solutions of equations based on standard Newton-Raphson's method and the secant method.
- A good starting value for Newton-Raphson's method seems to be γ/ρ.
- It is interesting to notice from the table that if $\rho << \alpha$ then $x^* \simeq \gamma/\rho$.

α	ρ	$-\lambda/a$	γ	γ/ρ	x^*
10	1	9	20	20	19.8896
10	1	9	5	5	4.8915
10	1	9	1	1	.9
10	9	1	10	1.1111	.7511
10	9	1	2.5	.2789	.0917
1	.5	.5	20	40	38.1592
1	.5	.5	5	10	8.4369

7. Some open questions and remarks

- Can this Theorem provide closed solutions for new examples of processes? (simple: work)

- We expect to find a closed formula from $g(x)$ to $\sigma(dy)$ for diffusions. (possible)

- Is it possible to extend this representation to finite horizon optimal stopping problems? (difficult ...)

- In particular: can this representation provide insights in the "smooth fitting" condition? (would be very nice, but ...)

Selected References

Boyarchenko, Levendorskii, *Nongaussian Merton Black Scholes Theorey*, Singapur, 2002.

Kyprianou,A. *Lectures in Lévy processes*, Springer, 2006.

Mordecki, E. *Optimal stopping and perpetual options for Lévy processes*, Finance and Stochastic (2002).

Mordecki, E., Salminen, P., *Otimal stopping for Hunt and Lévy processes*, Stochastics (2006).

Novikov, Shiryaev, *An effective solution to o.s.p for random walks*, Theory Probab. Appl. (2004)

Sato, K. *Levy Processes and Infinitely Divisible Distributions*, Cambrigde (1999).